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A SUSY approach for investigation of two-dimensional quantum mechanical systems*

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Abstract

Different ways to incorporate two-dimensional systems which are not amenable to separation of variables treatment into the framework of supersymmetrical quantum mechanics (SUSY QM) are analysed. In particular, the direct generalization of one-dimensional Witten SUSY QM is based on supercharges of first order in momenta and allows one to connect the eigenvalues and eigenfunctions of two scalar and one matrix Schrödinger operators. The use of second-order supercharges leads to polynomial supersymmetry and relates a pair of scalar Hamiltonians, giving a set of partner systems with almost coinciding spectra. This class of systems can be studied by means of a new method of SUSY separation of variables, where supercharges *allow* separation of variables, but Hamiltonians *do not*. The method of shape invariance is generalized to two-dimensional models in order to construct purely algebraically a chain of eigenstates and eigenvalues for generalized Morse potential models in two dimensions.

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1. Introduction

Supersymmetric quantum mechanics [1, 2] provides an interesting framework within which to analyse nonrelativistic quantal problems. In particular, it allows one to investigate the spectral properties of a wide class of quantum models and to generate new systems with given spectra. SUSY QM gives new insight into the problem of spectral equivalence of Hamiltonians, which, historically, has been constructed as a factorization method in quantum mechanics [3] and as Darboux–Crum transformations in mathematical physics [4].

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During the last two decades SUSY QM has become an important and popular tool for use in studying a wide variety of quantum systems (see the list of reports presented at this conference). It is easy to see that the main stream of the development in SUSY QM concerned one-dimensional models. Although the multi-dimensional (especially two- and three-dimensional) problems show much wider variety and are practically important, much less attention has been given in the literature to the study of these models in SUSY QM. Thus, future progress seems likely to be mostly connected with investigation of multi-dimensional generalizations. The main aim of this paper (based on a talk at the conference) is to summarize the different results of previous investigations of two-dimensional SUSY QM.

The paper is organized as follows. The two-dimensional generalization of the conventional Witten formulation of SUSY QM is formulated in section 2. In section 3 the two-dimensional SUSY QM with supercharges of second order in derivatives is presented. Section 4 contains a new, supersymmetric, approach for investigation of two-dimensional models, which *are not amenable* to separation of variables treatment. This method is based on the second-order supercharges introduced in the previous section, and it gives a new opportunity to reduce the problem to a one-dimensional one. Thus we obtain a specific method of SUSY *separation of variables*. Section 5 is devoted to the generalization of the well-known notion of shape invariance to two-dimensional models. Some new aspects typical for two-dimensional shape invariance are investigated. In section 6 the deformation of SUSY QM algebra for the models of sections 3–5 is described.

2. Direct two-dimensional generalization of the conventional Witten SUSY QM

The conventional one-dimensional SUSY QM was proposed by E Witten [1]. It is characterized by the simplest realization of SUSY algebra:

$$\{\hat{Q}^+, \hat{Q}^-\} = \hat{H} \qquad (\hat{Q}^+)^2 = (\hat{Q}^-)^2 = 0 \qquad [\hat{H}, \hat{Q}^\pm] = 0 \tag{1}$$

where the super-Hamiltonian $\hat{H}$ is a diagonal matrix $\hat{H} = \mathrm{diag}(h^{(0)}, h^{(1)})$, with $h^{(i)} = -\partial^2 + V^{(i)}(x), \partial \equiv d/\mathrm{d}x$, and supercharges $\hat{Q}^\pm$ are off diagonal with elements $q^\mp \equiv \pm\partial + \partial W(x)$ of first order in derivatives with superpotential $W(x)$. In terms of components, the (anti)commutation relations of SUSY algebra (1) mean, respectively, factorization of Hamiltonians, nilpotent structure of supercharges and intertwining of $h^{(i)}$ with $q^\pm$. The superpotential $W(x)$ is defined via an arbitrary (possibly non-normalizable) solution $\Psi(x) \equiv \exp(-W(x))$ of the Schrödinger equation $h^{(0)}\Psi(x) = \epsilon\Psi(x)$ with $\epsilon \leqslant E_0^{(0)}$. If this solution is nodeless, one has almost coinciding spectra of $h^{(0)}$ and $h^{(1)}$ or, equivalently, double degeneracy of the energy spectrum of H.

The direct multi-dimensional generalization of the construction above was built in [5] both by direct extension [6] of one-dimensional formulae and using [7] the superfield approach of quantum field theory. We call it 'direct' since it retains both all relations of superalgebra (1) and the first-order form of the components of the supercharges.

Here we restrict ourselves to the particular case of *two space dimensions* $\vec{x} = (x_1, x_2)$. The precise formulae for the 4×4 super-Hamiltonian and supercharges are then the following:

$$\hat{H} = \begin{pmatrix} h^{(0)}(\vec{x}) & 0 & 0 \\ 0 & h^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & h^{(2)}(\vec{x}) \end{pmatrix} \qquad i,k = 1,2 \qquad \hat{Q}^+ = (\hat{Q}^-)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q_1^- & 0 & 0 & 0 \\ q_2^- & 0 & 0 & 0 \\ 0 & p_1^+ & p_2^+ & 0 \end{pmatrix} \tag{2}$$

where two scalar Schrödinger operators $h^{(0)}$, $h^{(2)}$ and the 2×2 matrix Schrödinger operator $h^{(1)}_{ik}$ are expressed in *quasifactorized* form in terms of the components of supercharges:

$$h^{(0)} = q^+_l q^-_l = -\partial^2_l + V^{(0)}(\vec{x}) = -\partial^2_l + (\partial_l W(\vec{x}))^2 - \partial^2_l W(\vec{x}) \qquad \partial^2_l \equiv \partial^2_1 + \partial^2_2$$
$$h^{(2)} = p^+_l p^-_l = -\partial^2_l + V^{(2)}(\vec{x}) = -\partial^2_l + (\partial_l W(\vec{x}))^2 + \partial^2_l W(\vec{x})$$
$$h^{(1)}_{ik} = q^-_i q^+_k + p^-_i p^+_k = -\delta_{ik}\partial^2_l + \delta_{ik}((\partial_l W(\vec{x}))^2 - \partial^2_l W(\vec{x})) + 2\partial_i \partial_k W(\vec{x}).$$

The components of supercharges $q^\pm_l$, $p^\pm_l$ are again of first order and depend on the two-dimensional superpotential $W(\vec{x})$:

$$q^\pm_l \equiv \mp\partial_l + \partial_l W(\vec{x}) \qquad p^\pm_l \equiv \epsilon_{lk} q^\mp_k. \tag{3}$$

The two-dimensional 4×4 super-Hamiltonian and supercharges (2) realize the same conventional SUSY QM algebra (1). In components, the commutation relations in (1) are expressed as the intertwining relations between the matrix $h^{(1)}_{ik}$ and $h^{(0)}, h^{(2)}$ by operators $q^\pm_l$, $p^\pm_l$:

$$h^{(0)} q^+_i = q^+_k h^{(1)}_{ki} \qquad h^{(1)}_{ik} q^-_k = q^-_i h^{(0)} \qquad h^{(1)}_{ik} p^-_k = p^-_i h^{(2)} \qquad p^+_k h^{(1)}_{ki} = h^{(2)} p^+_i. \tag{4}$$

The energy spectra of $h^{(0)}$ and $h^{(2)}$ are in general different, but the intertwining relations (4) provide equivalence of energy spectra between a pair of scalar Hamiltonians $h^{(0)}$, $h^{(2)}$ and the 2×2 matrix Hamiltonian $h^{(1)}_{ik}$. 'Equivalence' means coincidence of spectra up to zero modes of operators $q^\pm_l$, $p^\pm_l$. Thus the supersymmetry (supersymmetric transformation) allows one to reduce the solution of the matrix Schrödinger problem with the Hamiltonian $h^{(1)}_{ik}$ to solution of a couple of scalar Schrödinger problems: those of $h^{(0)}$, $h^{(2)}$. Due to the same intertwining relations, the vector wavefunctions of the matrix Hamiltonian $h^{(1)}_{ik}$ are also connected (up to a normalization factor) with the scalar wavefunctions of the scalar Hamiltonians $h^{(0)}$, $h^{(2)}$:

$$\Psi^{(1)}_i(\vec{x}; E) = q^-_i \Psi^{(0)}(\vec{x}; E) \qquad i = 1, 2 \qquad \Psi^{(0)}(\vec{x}; E) = q^+_i \Psi^{(1)}_i(\vec{x}; E)$$
$$\Psi^{(1)}_i(\vec{x}; E) = p^-_i \Psi^{(2)}(\vec{x}; E) \qquad \Psi^{(2)}(\vec{x}; E) = p^+_i \Psi^{(1)}_i(\vec{x}; E).$$

The Schrödinger operators with a matrix potential are not particularly exotic in quantum mechanics. In particular, the above-described two-dimensional generalization of SUSY QM was successfully used [8] to investigate the spectra of Pauli operators for fermions in external electromagnetic fields. Nevertheless, considering this rather attractive construction, one question seems natural: is it possible to perform supersymmetric transformations in the two-dimensional case avoiding any matrix Hamiltonians?

3. Second-order supercharges in two-dimensional SUSY QM

The main idea, which could allow us to get rid of matrix components of the super-Hamiltonian, is to explore the supercharges of second order in derivatives. Such supercharges of higher orders in momenta were proposed for the one-dimensional situation for the first time in [9] (see also [10, 11]), leading to the polynomial deformation of SUSY algebra (see section 6). In general, this approach implies a deformation of one relation of SUSY algebra (1) only, namely that of the (quasi)factorization, but keeping unchanged the nilpotency of $\hat{Q}^\pm$ and the intertwining relations. The latter seem to be the most important ingredients of SUSY methods in QM.

The simplest variant of second-order supercharges $q^\pm$—of the so-called reducible [10] form—gives an uninteresting result in the two-dimensional case: the intertwined partner Hamiltonians differ by a trivial constant only, and both of them admit the separation of

variables (see the details in [12, 13]). For this reason, here we will be interested in general irreducible second-order components of supercharges:

$$q^+ = g_{ik}(\vec{x})\partial_i\partial_k + C_i(\vec{x})\partial_i + B(\vec{x}) \qquad q^- \equiv (q^+)^\dagger. \tag{5}$$

The important question that we have to investigate now concerns the existence of Hamiltonians

$$h^{(i)} = -\Delta^{(2)} + V^{(i)}(\vec{x}) \qquad i = 1, 2 \qquad \Delta^{(2)} \equiv \partial_l\partial_l \tag{6}$$

which satisfy the intertwining relations with $q^\pm$ of the form (5):

$$h^{(1)}q^+ = q^+h^{(2)} \qquad q^-h^{(1)} = h^{(2)}q^-. \tag{7}$$

The first consequence of (7) essentially restricts the possible 'metrics' $g_{ik}(\vec{x})$ via $\partial_l g_{ik} + \partial_i g_{lk} + \partial_k g_{il} = 0$ with solutions

$$\begin{aligned} &g_{11} = \alpha x_2^2 + a_1x_2 + b_1 \qquad g_{22} = \alpha x_1^2 + a_2x_1 + b_2 \\ &g_{12} = -\tfrac{1}{2}(2\alpha x_1x_2 + a_1x_1 + a_2x_2) + b_3. \end{aligned} \tag{8}$$

This has to be taken into account in rewriting [12] the intertwining relations (7) in components:

$$\partial_i C_k(\vec{x}) + \partial_k C_i(\vec{x}) + \Delta^{(2)} g_{ik}(\vec{x}) - (V^{(1)}(\vec{x}) - V^{(2)}(\vec{x}))g_{ik}(\vec{x}) = 0 \tag{9}$$

$$\Delta^{(2)} C_i(\vec{x}) + 2\partial_i B(\vec{x}) + 2g_{ik}(\vec{x})\partial_k V^{(2)}(\vec{x}) - (V^{(1)}(\vec{x}) - V^{(2)}(\vec{x}))C_i(\vec{x}) = 0 \tag{10}$$

$$\Delta^{(2)} B(\vec{x}) + g_{ik}(\vec{x})\partial_k\partial_i V^{(2)}(\vec{x}) + C_i(\vec{x})\partial_i V^{(2)}(\vec{x}) - (V^{(1)}(\vec{x}) - V^{(2)}(\vec{x}))B(\vec{x}) = 0. \tag{11}$$

The nonlinear system of second-order differential equations (9)–(11) for unknown functions $C_i(\vec{x})$, $B(\vec{x})$, $V^{(1)}(\vec{x})$, $V^{(2)}(\vec{x})$ with constant parameters α, a_i, b_i in $g_{ik}(\vec{x})$ does not admit a general solution, but one can look for particular solutions with concrete metrics g_{ik} and some ansätze for unknown functions C_i.

In particular, the system (9)–(11) is substantially simplified for the metrics of elliptic form $g_{ik}(\vec{x}) \equiv \delta_{ik}$. In this case all unknown functions in (9)–(11) can be found analytically in the general form [12], but for all such solutions both Hamiltonians $h^{(1)}, h^{(2)}$ turn out to admit the R separation [14] of variables in parabolic, elliptic or polar[1] coordinates; i.e. this class of two-dimensional problems can be reduced to two one-dimensional models.

A much more interesting situation appears for more complicated forms of metrics. Thus the list of particular solutions of the system (9)–(11) can be constructed analytically [12, 13, 15] for the hyperbolic metrics $g_{ik} = \mathrm{diag}(+1, -1)$. Indeed, for these metrics part of the task of solution of (9)–(11) can be carried out in a general form since these equations are reduced to a simpler system, which will be written in terms of coordinates $x_\pm \equiv x_1 \pm x_2$, $\partial_\pm \equiv \partial/\partial x_\pm$ and $C_+ \equiv C_1 - C_2$; $C_- \equiv C_1 + C_2$. The general solution can be provided by solving the system

$$\partial_-(C_-F) = -\partial_+(C_+F) \tag{12}$$

$$\partial_+^2 F = \partial_-^2 F \tag{13}$$

where $C_\pm$ depend only on $x_\pm$, respectively: $C_\pm \equiv C_\pm(x_\pm)$. The function F, the solution of (13), is represented as $F = F_1(x_+ + x_-) + F_2(x_+ - x_-)$. The potentials $V^{(1),(2)}(\vec{x})$ and the function $B(\vec{x})$ are expressed in terms of $F_1(2x_1)$, $F_2(2x_2)$ and $C_\pm(x_\pm)$, the solutions of (12):

$$\begin{aligned} &V^{(1),(2)} = \pm\tfrac{1}{2}(C_+' + C_-') + \tfrac{1}{8}\left(C_+^2 + C_-^2\right) + \tfrac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)) \\ &B = \tfrac{1}{4}(C_+C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)) \end{aligned} \tag{14}$$

where C' means the derivative with respect to its argument.

For lack of a regular procedure for solution of both equations of the system (12), (13), particular solutions can be found starting from certain ansätze for functions $C_\pm(x_\pm)$, $F(\vec{x})$.

[1] The reducible second-order supercharges just correspond to the separation of variables in polar coordinates.

(1) Let $C_- = 0$; then from (12) one obtains $F = \phi(x_-)/C_+(x_+)$. After inserting into equation (13) separation of variables is possible, and the particular solution reads[2]

$$C_+(x_+) = \frac{1}{\delta_1 \exp(\sqrt{\lambda}\cdot x_+) + \delta_2 \exp(-\sqrt{\lambda}\cdot x_+)}$$
$$F_{1,2}(2x) = \delta_1\sigma_{1,2}\exp(2\sqrt{\lambda}x) + \delta_2\sigma_{2,1}\exp(-2\sqrt{\lambda}x)$$

where the Greek letters are arbitrary constants; depending on the sign of λ, they may be real or complex.

(2) Let $F(\vec{x})$ allow also the factorization $F = F_+(x_+)\cdot F_-(x_-)$. Then, from equation (12),

$$C_\pm = \frac{\nu_\pm}{F_\pm} \pm \frac{\gamma}{F_\pm}\int^{x_\pm} F_\pm \,\mathrm{d}x'_\pm \tag{15}$$

and there are two options for fulfilling the condition (13), i.e. $F(\vec{x}) = F_1(2x_1) + F_2(2x_2)$:

$$(a)\quad F_\pm(x_\pm) = \epsilon_\pm x_\pm \qquad (b)\quad F_\pm = \sigma_\pm\exp(\sqrt{\lambda}\cdot x_\pm) + \delta_\pm\exp(-\sqrt{\lambda}\cdot x_\pm). \tag{16}$$

Corresponding potentials can be found according to equation (14), similar to the ones obtained in [16] in a quite different approach. Below, some other solutions of (12), (13) will be built up [15].

(3) Let us start now from the general solution of (12):

$$F = L\left(\int\frac{\mathrm{d}x_+}{C_+} - \int\frac{\mathrm{d}x_-}{C_-}\right)\Big/(C_+C_-). \tag{17}$$

Then equation (13) gives the functional differential equation for the functional $L(A_+ - A_-)$ with $A'_\pm \equiv 1/C_\pm(x_\pm)$:

$$\left(\frac{A'''_+}{A'_+} - \frac{A'''_-}{A'_-}\right)L(A_+ - A_-) + 3(A''_+ + A''_-)L'(A_+ - A_-) + (A'^2_+ - A'^2_-)L''(A_+ - A_-) = 0 \tag{18}$$

where L' denotes the derivative of L with respect to its argument. If we take functions $A_\pm$ such that $A''_\pm = \lambda^2 A_\pm$, $\lambda = \text{const}$, equation (18) will become an ordinary differential equation for L with independent variable $(A_+ - A_-)$. It can be easily solved:

$$L(A_+ - A_-) = \alpha(A_+ - A_-)^{-2} + \beta$$

where $A_\pm = \sigma_\pm\exp(\lambda x_\pm) + \delta_\pm\exp(-\lambda x_\pm)$ with $\sigma_+\cdot\delta_+ = \sigma_-\cdot\delta_-$ and α, β are real constants. For $\lambda^2 > 0$, choosing $\sigma_\pm = -\delta_\pm = k/2$ or $\sigma_\pm = +\delta_\pm = k/2$, we obtain (up to an arbitrary shift in $x_\pm$) two particular solutions:

$$(3a)\quad A_\pm = k\sinh(\lambda x_\pm) \qquad (3b)\quad A_\pm = k\cosh(\lambda x_\pm).$$

Then (17) leads to

$$\begin{aligned}(3a)\quad F_1(2x) &= \frac{k_1}{\cosh^2(\lambda x)} + k_2\cosh(2\lambda x)\\ F_2(2x) &= \frac{k_1}{\sinh^2(\lambda x)} + k_2\cosh(2\lambda x) \qquad C_\pm = \frac{k}{\cosh(\lambda x_\pm)} \qquad k\neq 0\end{aligned} \tag{19}$$

$$\begin{aligned}(3b)\quad F_1(2x) &= -F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2\sinh^2(\lambda x)\\ C_\pm &= \frac{k}{\sinh(\lambda x_\pm)} \qquad k\neq 0.\end{aligned} \tag{20}$$

For $\lambda^2 < 0$, hyperbolic functions must be replaced by trigonometric ones.

[2] Here and below, $F_{1,2}$ are defined only up to an arbitrary real constant: $F_1 \to F_1 + \epsilon$, $F_2 \to F_2 - \epsilon$.

We should remark that the case $\lambda^2 = 0$, i.e. $A''_\pm = 0$, is not of interest, leading to trivial superpartners. However, choosing in (20) $\lambda \to 0, k, k_1, k_2^{-1} \to 0$ simultaneously, so that $\lambda^2 \sim k_1 \sim k_2^{-1} \sim k^2$, we obtain the solution

$$F_1(2x) = -F_2(2x) = \tilde{k}_1 x^{-2} + \tilde{k}_2 x^2 \qquad C_\pm = \frac{\tilde{k}}{x_\pm}. \tag{21}$$

One can check that (12) is also satisfied by

$$F_1(2x) = -F_2(2x) = k_1 x^2 + k_2 x^4 \qquad C_\pm = \pm\frac{k}{x_\pm}. \tag{22}$$

(4) Starting again from (17), it is convenient to pass on to new variable functions $C_\pm \equiv \pm f_\pm / f'_\pm$. Then F in (17) is represented in the form $F = U(f_+ f_-) f'_+ f'_-$ with an arbitrary[3] function U. After substitution in (13) one obtains the functional differential equation

$$\left(f'^2_+ f^2_- - f^2_+ f'^2_-\right) U''(f) + 3f\left(\frac{f''_+}{f_+} - \frac{f''_-}{f_-}\right) U'(f) + \left(\frac{f'''_+}{f'_+} - \frac{f'''_-}{f'_-}\right) U(f) = 0$$
$$f \equiv f_+ f_-.$$

For a particular form of functions $f_\pm = \alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm)$, this equation becomes an ordinary differential equation for U with independent variable f. Its solution is $U = a + 4bf_+ f_-$ (a, b real constants). Then functions

$$\begin{aligned} F_1(x) &= k_1(\alpha_+\alpha_- \exp(\lambda x) + \beta_+\beta_- \exp(-\lambda x)) + k_2\left(\alpha_+^2\alpha_-^2 \exp(2\lambda x) + \beta_+^2\beta_-^2 \exp(-2\lambda x)\right) \\ -F_2(x) &= k_1(\alpha_+\beta_- \exp(\lambda x) + \beta_+\alpha_- \exp(-\lambda x)) + k_2\left(\alpha_+^2\beta_-^2 \exp(2\lambda x) + \beta_+^2\alpha_-^2 \exp(-2\lambda x)\right) \\ C_\pm &= \pm\frac{\alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm)}{\lambda(\alpha_\pm \exp(\lambda x_\pm) - \beta_\pm \exp(-\lambda x_\pm))} \end{aligned} \tag{23}$$

(with $k_1 \equiv a\lambda^2, k_2 \equiv 4b\lambda^2$) are real solutions of (12), (13) if $\alpha_\pm, \beta_\pm$ are real for $\lambda^2 > 0$, and $\alpha_\pm = \beta_\pm^*$ for $\lambda^2 < 0$.

(5) To find the next class of solutions it is useful to rewrite (12) in terms of $x_{1,2}$:

$$(F_1(2x_1) + F_2(2x_2))\partial_1(C_+ + C_-) + F'_1(2x_1)(C_+ + C_-) + F'_2(2x_2)(C_+ - C_-) = 0.$$

Among the known particular solutions the most compact one is

$$C_+(x) = C_-(x) = ax^2 + c \qquad F_1(2x_1) = 0 \qquad F_2(2x_2) = \frac{b^2}{x_2^2}. \tag{24}$$

After inserting these solutions (19)–(24) into the general formulae (14), one obtains the analytical expressions for the potentials. Their explicit form can be found in [15].

The additional class of particular solutions of the system (9)–(11) obtained for the case of degenerate metrics $g_{ik} = \mathrm{diag}(1, 0)$ can be found also in [15].

4. SUSY separation of variables

From the very beginning of this paper we have been interested in two-dimensional quantum systems, which *are not amenable* to separation of variables treatment. The supersymmetric approach, namely that via the intertwining relations (7), allows one to formulate a specific supersymmetric alternative to the conventional notion of separation (including the so-called R separation [14]) of variables. The main idea [17] is to consider a particular class of solutions

[3] Due to equation (13), the function F should be additionally representable in the form $F = F_1(2x_1) + F_2(2x_2)$.

of intertwining relations (7) when the components of the supercharge $q^{\pm}$ *are amenable* to separation of variables treatment but Hamiltonians $h^{(i)}$ *are not*. In this case the Hamiltonians $h^{(i)}$ turn out to be partially solvable or, in other terminology, quasi-exactly solvable [18]. Both terms above mean that parts of the spectrum (and possibly some of the corresponding eigenfunctions) of the Hamiltonian are known. The crucial ingredient of the approach [17] is in the investigation of zero modes of intertwining operators $q^{\pm}$.

The general scheme of the method is the following. Let us suppose that $N+1$ normalizable zero modes of q^+ are known (for example, due to separation of variables in q^+):

$$q^+\Omega_n(\vec{x}) = 0 \qquad n = 0, 1, \ldots, N \qquad q^+\vec{\Omega}(\vec{x}) = 0 \tag{25}$$

where $\vec{\Omega}(\vec{x})$ is a column vector with components $\Omega_n(\vec{x})$. From the intertwining relations (7) one can see that the space of zero modes is closed under the action of $h^{(2)}$, and therefore

$$h^{(2)}\vec{\Omega}(\vec{x}) = \hat{C}\vec{\Omega}(\vec{x}) \tag{26}$$

where $\hat{C} \equiv \|c_{ik}\|$ is a c-number $\vec{x}$-independent real matrix. If the matrix $\hat{C}$ can be diagonalized by a real similarity transformation:

$$\hat{B}\hat{C}(\hat{B})^{-1} = \hat{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N) \tag{27}$$

the problem is reduced to a standard algebraic task within the zero-mode space:

$$h^{(2)}(\hat{B}\vec{\Omega}(\vec{x})) = \hat{\Lambda}(\hat{B}\vec{\Omega}(\vec{x})). \tag{28}$$

It is not clear in advance whether this general scheme is realized practically. To put this differently, are there any solutions $C_i(\vec{x})$, $B(\vec{x})$ of the intertwining relations (9)–(11) which give q^+ with separation of variables?

To investigate this problem, it is useful [17] to transform the supercharge q^+ by the special similarity transformation, which removes the terms linear in derivatives:

$$\begin{aligned} \tilde{q}^+ &= \mathrm{e}^{(-\chi(\vec{x}))}q^+\,\mathrm{e}^{(+\chi(\vec{x}))} = \partial_1^2 - \partial_2^2 + \frac{1}{4}(F_1(2x_1) + F_2(2x_2)) \\ \chi(\vec{x}) &= -\frac{1}{4}\left(\int C_+(x_+)\,\mathrm{d}x_+ + \int C_-(x_-)\,\mathrm{d}x_-\right). \end{aligned} \tag{29}$$

These new operators $\tilde{q}^+$ obviously obey the condition of separation of variables realizing the first step of our scheme of SUSY separation of variables. Zero modes of $\tilde{q}^+$ can be found as linear superpositions of products of one-dimensional wavefunctions $\eta_n(x_1)$ and $\rho_n(x_2)$, satisfying Schrödinger equations (with ϵ_n the separation constants):

$$\left(-\partial_1^2 - \tfrac{1}{4}F_1(2x_1)\right)\eta_n(x_1) = \epsilon_n\eta_n(x_1) \qquad \left(-\partial_2^2 + \tfrac{1}{4}F_2(2x_2)\right)\rho_n(x_2) = \epsilon_n\rho_n(x_2). \tag{30}$$

In analogy to (29), one can define operators

$$\tilde{h} \equiv \exp(-\chi(\vec{x}))h^{(2)}\exp(+\chi(\vec{x})) = -\partial_l^2 + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2 - \tfrac{1}{4}F_1(2x_1) + \tfrac{1}{4}F_2(2x_2) \tag{31}$$

and eigenfunctions of $\tilde{q}^+$ as

$$\tilde{\Omega}_n(\vec{x}) = \exp(-\chi(\vec{x}))\cdot\Omega_n(\vec{x}) \tag{32}$$

keeping in mind however that the normalizability and orthogonality are not preserved automatically due to non-unitarity of the similarity transformation.

Then using (30) one can write

$$\tilde{h}\tilde{\Omega}_n(\vec{x}) = [2\epsilon_n + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2]\tilde{\Omega}_n(\vec{x}). \tag{33}$$

It is not evident from (33), but the space spanned by functions $\tilde{\Omega}_n(\vec{x})$ is closed under the action of $\tilde{h}$. This will be demonstrated explicitly in the concrete model below.

In contrast to (29), where variables are separated, no separation for $\tilde{h}$, which would make the two-dimensional dynamics non-trivially reducible to one-dimensional dynamics. In view

of this, we refer to this method [17] for achieving partial solvability as *SUSY separation of variables*.

The scheme of SUSY separation of variables formulated above can be used for arbitrary models satisfying the intertwining relations (7). The list of such models is already rather long, and it may increase in the future, but it is very important to check the applicability of the scheme to the concrete model where the explicit solutions can be constructed. Actually, this means that solutions of two one-dimensional problems (30) can be found analytically. Below we briefly describe such a model—the generalized two-dimensional Morse potential.

Among the solutions [15] of the system (12), (13) we focus attention on the particular case with a specific choice of parameters and $A > 0, \alpha > 0, a$ real constants[4]:

$$C_+ = 4a\alpha \qquad C_- = 4a\alpha \cdot \coth\frac{\alpha x_-}{2} \tag{34}$$

$$f_i(x_i) \equiv \frac{1}{4}F_i(2x_i) = -A(\mathrm{e}^{-2\alpha x_i} - 2\,\mathrm{e}^{-\alpha x_i}) \qquad i = 1, 2 \tag{35}$$

$$V^{(1),(2)} = \alpha^2 a(2a \mp 1)\sinh^{-2}\left(\frac{\alpha x_-}{2}\right) + 4a^2\alpha^2 + A[\mathrm{e}^{-2\alpha x_1} - 2\,\mathrm{e}^{-\alpha x_1} + \mathrm{e}^{-2\alpha x_2} - 2\,\mathrm{e}^{-\alpha x_2}]. \tag{36}$$

One easily recognizes in (36) a sum of two Morse potentials plus a hyperbolic singular term which prevents one for applying the *conventional* methods of separation of variables. These singular terms can be both attractive, for $|a| > \frac{1}{2}$, or one repulsive and one attractive, for $|a| < \frac{1}{2}$. The parameter a will be further constrained by the condition that the strength of the attractive singularity at $x_- \to 0$ should not exceed the well-known bound $-1/(4x_-^2)$.

The normalizable functions $\tilde{\Omega}_n(\vec{x})$ (and $\Omega_n(\vec{x})$) can be constructed from the well known [20] normalizable solutions of (30) with $\epsilon_n < 0$:

$$\tilde{\Omega}_n(\vec{x}) = \exp\left(-\frac{\xi_1 + \xi_2}{2}\right)(\xi_1\xi_2)^{s_n} F(-n, 2s_n + 1; \xi_1)F(-n, 2s_n + 1; \xi_2) \tag{37}$$

where $F(-n, 2s_n + 1; \xi)$ is the standard degenerate (confluent) hypergeometric function, reducing to a polynomial for integer n, and

$$\xi_i \equiv \frac{2\sqrt{A}}{\alpha}\exp(-\alpha x_i) \qquad s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > 0 \qquad \epsilon_n = -A\left[1 - \frac{\alpha}{\sqrt{A}}\left(n + \frac{1}{2}\right)\right]^2. \tag{38}$$

The number $(N + 1)$ of normalizable zero modes (37) is determined by the inequality $s_n > 0$.

The condition of normalizability of zero modes $\tilde{\Omega}_n(\vec{x})$, together with the absence of the 'fall to the centre', leads [17] to the following two ranges of parameters:

$$a \in \left(-\infty, -\frac{1}{4} - \frac{1}{4\sqrt{2}}\right) \qquad s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > -2a > 0. \tag{39}$$

$$a \in \left(-\frac{1}{4}, \frac{1}{4}\right) \qquad s_0 > 2(|a| + 1). \tag{40}$$

In sections 4 and 5, only the region (39) will be considered. Inequalities (39) can be satisfied by the choice of a and A, and/or by suitable restriction on the number N of zero modes $\Omega_n(\vec{x})$.

Analysis of the action of $\tilde{h}$ in (33) gives that the matrix $\hat{C}$ is of tridiagonal form. It can be diagonalized explicitly by a similarity transformation, and the eigenvalues E_k of $h^{(2)}$ coincide with its (all different and nonzero) diagonal elements:

$$E_k = c_{kk} = -2(2a\alpha^2 s_k - \epsilon_k). \tag{41}$$

[4] For the complexification of the model, see [19].

The resulting eigenfunctions of $h^{(2)}$ are obtained (see equations (27), (28)) from the constructed zero modes $\tilde{\Omega}_n(\vec{x})$ and the similarity transformation $\hat{B}$:

$$\Psi_{N-n}(\vec{x}) = \sum_{l=0}^{N} b_{nl}\Omega_l(\vec{x}). \tag{42}$$

For the algorithm for iterative construction of coefficients b_{nl}, see [17]. Thus the construction of the set of eigenfunctions, which lie in the space of zero modes, is completed.

These eigenfunctions $\Psi_k(\vec{x})$ may also be used for constructing more general eigenfunctions of $h^{(2)}$ via a product ansatz:

$$\Phi(\vec{x}) \equiv \Psi_k(\vec{x}) \cdot \Theta(\vec{x}). \tag{43}$$

Three such eigenfunctions based on Ψ_0 were constructed in [17]. Within the bounds imposed (39), only one of them is normalizable, though for the region (40) all three are normalizable.

5. Shape invariance in two dimensions

In the previous section we developed a method which led to construction of the partially solvable (quasi-exactly solvable) two-dimensional models. Let us recall now the well known *in one dimension* and very elegant method of *shape invariance* [21] usually associated with the exactly solvable one-dimensional systems. Our aim here is to generalize the idea of shape invariance to the two-dimensional case [17].

Referring readers to the original paper [21] and reviews [2] for the detailed discussion of the standard one-dimensional shape invariance method, let us list its main steps only:

$$\tilde{H}(x;a) = H(x;\bar{a}) + \mathcal{R}(a) \qquad \bar{a} = \bar{a}(a) \tag{44}$$

where $\mathcal{R}(a)$ is a (c-number) function of a. The absence of spontaneous breaking of supersymmetry for all values of a implies that the lowest eigenvalue $E_0(a)$ of $H(a)$ vanishes and the corresponding eigenfunctions $\Psi_0(a)$ are normalizable zero modes of $Q^+(a)$.

The intertwining relations $Q^-(x;a)\tilde{H}(x;a) = H(x;a)Q^-(x;a)$ with the standard first-order supercharge allow one in this case to solve the entire spectral problem for $H(x;a)$. The crucial steps are as follows:

$$H(x;\bar{a})\Psi_0(x;\bar{a}) = E_0(\bar{a})\Psi_0(x;\bar{a}) = 0 \qquad \tilde{H}(x;a)\Psi_0(x;\bar{a}) = \mathcal{R}(a)\Psi_0(x;\bar{a}). \tag{45}$$

It is important to note that $\Psi_0(x;\bar{a}) \equiv \tilde{\Psi}_0(x;a)$ has no nodes and therefore is the ground-state wavefunction of $\tilde{H}(x;a)$. Then

$$H(x;a)[Q^-(x;a)\Psi_0(x;\bar{a})] = \mathcal{R}(a)[Q^-(x;a)\Psi_0(x;\bar{a})]. \tag{46}$$

Provided that $[Q^-(xa)\Psi_0(x;\bar{a})]$ is normalizable, we have generated an excited state of $H(x;a)$, and thus $\mathcal{R}(a)$ is naturally positive. It is clear that these steps can be repeated up to the last one, where the resulting wavefunction Ψ will no longer be normalizable.

It is also clear that the isospectrality of $H(x;a)$ and $\tilde{H}(x;a)$ (up to the only zero mode $\Psi_0(x;a)$) implies that there is no eigenvalue of $H(x;a)$ between zero and the ground-state energy $\tilde{E}_0(a)$ of $\tilde{H}$. This observation leads to a proof that after suitable iterations one gets the entire spectrum of $H(x;a)$. This method is referred to as algebraic solvability (or complete solvability) by means of shape invariance in one-dimensional SUSY QM.

To proceed to the formulation of the two-dimensional shape invariance method, we start from the relatively simple two-dimensional case of systems with conventional separation of variables:

$$H(\vec{x}) = H_1(x_1) + H_2(x_2) \qquad \vec{x} = (x_1, x_2).$$

Now suppose that H_1 and H_2 both are shape invariant:

$$\tilde{H}_i(x_i;a_i) = H_i(x_i;\bar{a}_i) + \mathcal{R}_i(a_i) \quad \leftrightarrow \quad \tilde{H}(\vec{x};\mathbf{a}) = H(\vec{x};\bar{\mathbf{a}}) + \mathcal{R}(\mathbf{a});\ \mathbf{a} \equiv (a_1,a_2). \tag{47}$$

In order to realize non-trivial intertwining relations for H, $\tilde{H}$ one can consider factorized supercharges of second order written as products of first-order supercharges:

$$Q^{\pm} = Q_1^{\pm}\cdot Q_2^{\pm} \qquad Q_i^{\pm} = \mp\partial_i + W_i(x_i). \tag{48}$$

There is the considerable difference from the one-dimensional case. The crucial reason is that the space of zero modes of supercharges becomes now of higher dimensionality, including the products of one-dimensional zero modes of the first Hamiltonian times all states of the second Hamiltonian and vice versa. While iterations are again obviously possible, it is clear that one cannot argue about the entire solvability of the spectral problem, because in general many zero modes of (48) exist. Their number depends on the confining properties of H_1 and H_2. For example, in a case of oscillator-like potentials this number becomes infinite, and they are distributed over the whole spectrum. In this case only partial solvability of H can be achieved via the choice of (48) and shape invariance. Of course, one can solve such trivial models by separate use of $Q^{\pm} = Q_i^{\pm}$, which allows one to solve for the entire spectrum of the two-dimensional model in terms of one-dimensional ones.

Let us suppose that we have a two-dimensional system (without separation of variables) with a Hamiltonian H, which is related to $\tilde{H}$ by (44). For simplicity (in general, there is no connection between the dimensionality of the Schrödinger equation and the dimensionality of the parameter manifold), we assume that shape invariance is realized with one parameter a. Two-dimensional SUSY QM does not identify here zero modes of $Q^{\pm}$ with the ground state of the Hamiltonian. Thus one has to repeat the steps (44)–(46) above, taking into account $E_0(a) \neq 0$. In order to make our discussion more explicit, we will from now on refer explicitly to the model (34)–(36) ($H \equiv h^{(2)}$), $\tilde{H} \equiv h^{(1)}$ with the parameter a being bound by (39).

First of all we observe that this model is indeed shape invariant (the infinite domain given by (39) allows one iteration of (44)):

$$\bar{a} = a - \tfrac{1}{2} \qquad \mathcal{R}(a) = \alpha^2(4a-1). \tag{49}$$

The starting point is to write (46)

$$H(\vec{x};a)\big[Q^-(\vec{x};a)\Psi_0\big(\vec{x};a-\tfrac{1}{2}\big)\big] = \big(E_0\big(a-\tfrac{1}{2}\big) + \mathcal{R}(a)\big)\cdot\big[Q^-(\vec{x};a)\Psi_0\big(\vec{x};a-\tfrac{1}{2}\big)\big] \tag{50}$$

where $E_0(a)$ and $\Psi_0(\vec{x};a)$ are not to be identified with the ground state. Thus we have constructed the new eigenstate and eigenvalue of $H(\vec{x};a)$, provided that $Q^-(\vec{x};a)\Psi_0\big(\vec{x};a-\frac{1}{2}\big)$ is normalizable. Note that the eigenvalue $\big(E_0\big(a-\frac{1}{2}\big)+\mathcal{R}(a)\big)$ is larger than $E_0(a)$ with the bounds of (39).

It is interesting that the energy of the first iteration of the shape invariance method in (50) coincides precisely with the eigenvalue of the additional solution $\Phi(\vec{x})$ mentioned in the previous section:

$$E = E_0\big(a-\tfrac{1}{2}\big) + \mathcal{R}(a) = \alpha^2[4a(1-s_0) + (2s_0-1)] + 2\epsilon_0.$$

The next iteration of the shape invariance method will give

$$H(a)\big[Q^-(a)Q^-\big(a-\tfrac{1}{2}\big)\Psi_0(a-1)\big] = \big(E_0(a-1) + \mathcal{R}\big(a-\tfrac{1}{2}\big) + \mathcal{R}(a)\big)\cdot\big[Q^-(a)Q^-\big(a-\tfrac{1}{2}\big)\Psi_0(a-1)\big] \tag{51}$$

and the new eigenfunction $Q^-(a)Q^-\big(a-\frac{1}{2}\big)\Psi_0(a-1)$ can be written explicitly as a function of $\vec{x}$. Provided that normalizability is ensured, one can thereby construct a chain by successive iterations of (50) and (51), since $Q^-(a)$ has no normalizable zero modes in (39). The end point of such a chain will be given by non-normalizability of the relevant wavefunction.

Let us stress that although we illustrated both methods using the same model, SUSY separation of variables can be implemented completely independently from shape invariance. For example, the model considered in the range (40) admits the method of SUSY separation of variables, but the shape invariance method cannot be applied since the domain (40) is too small.

6. Polynomial algebra for two-dimensional SUSY QM

In this last section we will analyse the deformation of conventional SUSY QM algebra (1) due to the introduction of second-order components $q^{\pm}$ of supercharges in sections 3–5. It is obvious that, keeping the intertwining relations (7) and the matrix structure of supercharges $\hat{Q}^{\pm}$, we do not change two relations of SUSY algebra (1) which express the supersymmetry of $\hat{H}$ and nilpotency of $\hat{Q}^{\pm}$. But the third relation, which gives the (quasi)factorization of the components of $\hat{H}$, cannot be fulfilled. In the one-dimensional case the anticommutator of $\hat{Q}^{\pm}$ gives [10] the operator of fourth order in derivatives which can be represented as a second-order polynomial of the super-Hamiltonian $\hat{H}$. The situation changes crucially for the two-dimensional systems of sections 3–6, where in general a new *diagonal* operator of fourth order appears:

$$\hat{R} \equiv \{\hat{Q}^{+}, \hat{Q}^{-}\}. \tag{52}$$

This operator obviously commutes with the super-Hamiltonian $\hat{H}$ due to the supersymmetry of $\hat{H}$ (intertwining relations (7)). It is shown in [13] that for Laplacian metrics in (5) $g_{ik} = \delta_{ik}$ (where variables can be separated), the operator $\hat{R}$ can be reduced to the second-order symmetry operator $\tilde{R}$ up to a second-order polynomial (with constant coefficients) in $\hat{H}$. But for all other metrics g_{ik}, including hyperbolic and degenerate ones, the following theorem was proved [13]: *the symmetry operator $\hat{R}$ is essentially of fourth order in derivatives; its order cannot be reduced*. The components $R^{(i)}$, $i = 1, 2$, of $\hat{R}$ are the symmetry operators of the components $h^{(i)}$: $[h^{(i)}, R^{(i)}] = 0$. Explicit expressions for $R^{(i)}$ for known solutions of (9)–(11) can be written out (see [22]). Thus all two-dimensional models of section 3 are *integrable*[5].

As for the generalized Morse model of section 4, the quantum integral of motion $R = Q^{-}Q^{+}$ gives zero for the eigenfunctions $\Psi_k(\vec{x})$ by construction, since they are zero modes of Q^{+}. But by a direct calculation, one can check that all three additional eigenfunctions $\Phi(\vec{x})$ (see (43)) of H are simultaneously [17] eigenfunctions (with nonzero eigenvalues) of the symmetry operator R. Thus they belong to a system of common eigenfunctions of two Hermitian mutually commuting operators H and R.

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